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Overview of a Methodology for Scaling the Indeterminate Equations of Wall Turbulence

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Recent efforts by the present authors have focused on the fundamental multiscaling behaviors of the time averaged dynamical equations of wall turbulence. These efforts have generated a number of new results relating to dynamical structure, as well as a new mathematical foundation. Central to this has been the development of the so-called method of scaling patches. This method provides a formalism for determining scaling behaviors directly from the indeterminate equations. A general description of this methodology is provided herein, and in doing so its connections to well-established scaling notions are identified. Example problems for which the method has been successfully applied includes turbulent boundary layer, pipe and channel flows, turbulent Couette–Poiseuille flow, fully developed turbulent heat transfer in a channel, and favorable pressure gradient boundary layers.

Nomenclature

R	=	Reynolds number
r_σ	=	ratio of the local mean velocity to the bulk mean velocity
T	=	Reynolds shear stress, $T = -\rho\overline{u'v'}$
T_θ	=	normalized turbulent heat flux
U	=	mean axial velocity component
U_∞	=	freestream velocity
u	=	velocity component in the x direction
u_τ	=	friction velocity, $u_\tau = \sqrt{\tau_w/\rho}$
v	=	velocity component in the y direction
x	=	streamwise coordinate direction
y	=	wall-normal coordinate direction
β	=	scale hierarchy parameter
δ	=	boundary layer thickness, channel half-height
ϵ	=	small parameter, $\epsilon^2 = 1/\delta^+$
η	=	outer-normalized distance from the wall, $\eta = y/\delta$
ν	=	kinematic viscosity
ρ	=	mass density
σ	=	parameter associated with the maximum inner-normalized temperature difference
τ_w	=	mean wall shear stress
Ψ	=	normalized temperature difference
ω	=	ratio of the wall shear stresses in turbulent Couette–Poiseuille flow

Subscript

m	=	denotes maximum value

Superscript

$+$	=	denotes inner normalization
$*$	=	denotes a general normalization
$'$	=	denotes fluctuation about the mean

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I. Introduction

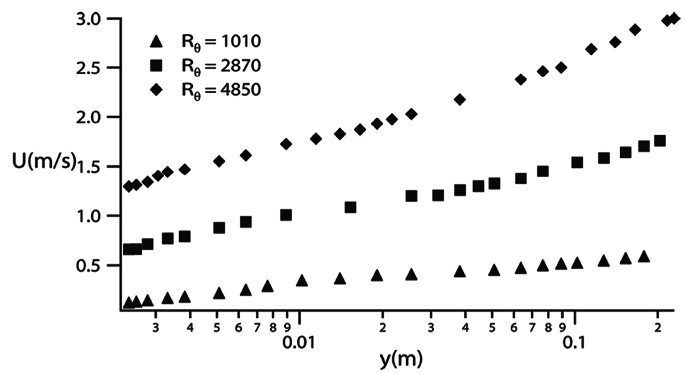
THE present effort describes a methodology for determining the scaling behaviors of the time averaged differential equations associated with wall-bounded turbulent flows. Herein scaling will refer not just to any nondimensionalization, but also to those that render the profiles of the variables of interest invariant over some portion of the flow domain as the relevant parameter(s), for example, Reynolds number, are varied. Another restriction comes from the observation that if the length scale is chosen too small, the resulting scaled variable functions will appear constant. Therefore, only those scalings under which at least one of the scaled profiles exhibit nontrivial variation will be of interest.

Similar to their laminar counterparts, the mean equations for wall-bounded flows along with their boundary conditions pose a singular perturbation type problem when the Reynolds number is large. The primary complication in determining scaling properties from such equations, however, lies in the fact that they are underdetermined. This fact arises owing to the Reynolds stress gradient term that appears as a result of time averaging. The effect of time averaging is, of course, at the heart of the turbulence closure problem, and with regard to scaling, it is rational to expect this averaging process to obscure and limit what might in principle be derivable from analysis of the governing equations alone.

Given these considerations, ongoing studies have been devoted toward elucidating what the time averaged equations themselves admit with regard to scaling, as well as the associated implications pertaining to flow physics [1–5]. In this regard, the objectives of the present paper are to: 1) provide an exposition of the rationale for, and overview of, the underlying methodology (the method of *scaling patches*) that has emerged from these efforts, and 2) provide an accounting of some of the scalings that have been determined and the flow physics that have been revealed.

To provide an appropriate context, it is useful to briefly review the predominant methodologies employed with regard to scaling wall-flow statistics. These are broadly characterized as mixing length, similarity, and overlapping layer arguments.

1) Mixing length-type arguments [6–8] rely on a phenomenologically based assumption that within the flow there exists a continuous hierarchy of length scales. The complementary assumption that these mixing lengths are proportional to the distance from the wall is often made as well. As originally conceived, the mixing length concept is devoid of any connection with the mean momentum balance (MMB).



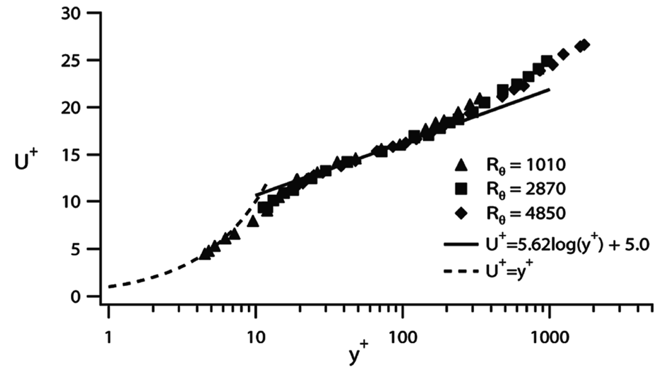


Fig. 1 Dimensional a) and innernormalized b) mean velocity profiles in turbulent boundary layers.

2) Similarity arguments (complete or incomplete) [9,10] rely on essentially ad hoc assumptions about the general character of certain unknown dimensionless functions of dimensionless variables as one of the latter (at least) approaches infinity. Similarity arguments of this type do not rely upon or employ the MMB, with the exception that in the dimensional analysis the unknown functions must be dimensionally consistent with this equation.

3) Overlapping layer arguments rely on the assumptions that there exists an intermediate domain between the inner and outer regions within which the mean velocity gradient can be validly scaled by either inner or outer normalizations [11], and that within this overlap region the mean velocity is increasing. This pair of assertions forms a particularly strong assumption, as can be shown by generic counterexamples [4]. Overlap layer based analyses do not employ the MMB except possibly in the process of postulating the existence of the traditional inner or outer scalings. In this regard, however, a rigorously grounded prediction of these scales is seldom made.

The present methodology is distinct from the mixing length, similarity, and overlap layer approaches in a number of ways. Important among these is that it avoids the assumptions stated under each bullet listed above. Similarly, in contrast to the approaches listed above, essential use is made of the MMB. Specifically, the approach is based upon a systematic method for locating the scaling patches, that is, regions of the flow together with the scalings appropriate for each, admitted by the MMB. The methodology is based upon the assumed *admissibility* and *compatibility* criteria discussed in Sec. II below. As distinguished from the assumptions of the approaches listed above, these assumed criteria broadly relate to guiding the methodology in determining which normalizations are admitted by the MMB as scalings, and what it physically means to have a balance equation of any kind. Owing to the indeterminate nature of the MMB for turbulent wall flows, experimental and numerical data have traditionally played a role, and within the scaling patch methodology continue to do so. In this regard, however, data are used, for example, to verify whether the *nominal* order of magnitude of a term under a certain normalization is the same as its *actual* order of magnitude, or whether certain scaled derivatives are of the expected order of magnitude. Perhaps the clearest example of the former of these is given in Wei et al. [1] in which all three terms in the MMB are empirically verified to have the same order of magnitude within the mesolayer. Examples of the latter may be found in Fife et al. [3]. Important elements of these uses of experimental data are that they guide the analysis rather than simply verify the end result, and in most instances encountered thus far the measurement accuracy requirements are well within experimentally attainable uncertainties.

Experimentally based studies exploring, for example, Reynolds number dependence employ a well-established procedure for testing candidate scaling(s) for statistical profiles across the flow. Given this, the following recapitulation and examination of the implications of this empirical test for assessing the validity of any given normalization (existence/nonexistence of a particular scaling) may be useful.

Consider the typical case in which wall-normal profiles of a velocity field statistic, say $\phi(y)$, are acquired over a range of Reynolds numbers. In their dimensional form, these measured statistical profiles (empirically determined functions) will generally vary widely in their magnitude and shape as the Reynolds number is varied. For each point in the profile, the statistic and the y value are made nondimensional according to the normalization being tested, for example, inner or outer. (The conventional inner normalization involves the use of ν, and the friction velocity $u_\tau = \sqrt{\tau_w/\rho}$. Similarly, outer normalization makes use of U_∞ and some measure of the overall thickness of the flow like δ.) If the different Reynolds number profiles (or more likely a portion of the profiles) merge to a single curve under the given normalization, and that curve exhibits nontrivial variation (is not flat), then this provides evidence that the particular normalization constitutes a scaling over the indicated subdomain. An extensively studied example of this is shown in Figs. 1a and 1b which compare dimensional and innernormalized mean velocity profile data for a zero-pressure-gradient turbulent boundary layer. In this case, the data convincingly merge to a single curve over a near-wall zone. Of course, the empirical test alone can never *prove* the existence of a scaling or its precise domain of validity, because all measurements are prone to uncertainties and all experiments are conducted for finite parameter values. This inherent limitation to the purely empirical study of turbulent wall flows points to the desirability of a guiding theoretical framework that retains a direct connection with the governing equations.

Nevertheless, this example highlights that a successful normalization (i.e., scaling) must stretch or compress the statistical function and its y value such that the normalized function $\phi^*(y^*)$ is rendered invariant with changes in Reynolds number or other parameters. For example, Fig. 1b provides evidence that under inner normalization and over the indicated Reynolds number range the scaled distribution of the mean velocity $U^+(y^+)$ is invariant with Reynolds number over a portion of the flow domain near the wall.

At a minimum, attaining such a situation requires that both the normalized values of ϕ and the normalized values of y respectively retain their same order of magnitude as the Reynolds number is varied. This said, it is relevant to note that satisfying this criterion does not necessarily guarantee that the normalized profiles will merge to a single invariant curve. If this criterion is not satisfied, however, it is incontrovertibly true that the profiles will fail to merge. Thus, the empirical test can provide convincing proof that a normalization does not constitute a scaling (i.e., those normalizations that fail to merge differing Reynolds number data with the differences being well beyond the uncertainty of the measurements).

The scaling patch methodology described below involves the recognition that the statistical profile under consideration can be identified as a solution to a rescaled averaged transport equation, and that, in fact, its scaling properties can be determined through an analysis of this equation. The basic averaged physical law of fluid dynamics, upon which all the considerations below are grounded, is the mean momentum balance obtained by Reynolds averaging the Navier–Stokes equations. The analysis methodology described herein employs the notion of scaling as a linear, parameter dependent transformation of the independent and dependent variables that results in an invariant form of the MMB that is valid over some portion of the flow domain. Such a notion, of course, has a long history of use in laminar flow theory where the equations are closed. In this regard, the present methodology can be considered a framework for rationally applying this concept to indeterminate equations.

II. Formalization of the Scaling Framework

First of all, it should be noted that the discussion will involve orders of magnitude of various quantities and their derivatives. These orders of magnitude are functions of R, and therefore their knowledge conveys information about how certain quantities grow or decay as $R \to \infty$. Actual orders of magnitude of physical quantities must be distinguished conceptually from the nominal orders of magnitude of those quantities when they appear as terms in some version of the averaged transport equation. Each such term is the rescaled derivative of some flow quantity times a coefficient. The nominal order of magnitude of such a term is simply the order of magnitude of that coefficient, that is, that which the term would have if the derivative were of order of magnitude unity. The methodology is to establish criteria by which one may surmise the existence of "scaling patches," and then conduct a search for them. The search for scaling patches, and their associated "admissible differential" scalings will now be described (a differential scaling is a scaling applied to differentials of the independent and dependent variables).

Succinctly, a scaling patch is a scaling, together with a y subdomain over which the normalized variables and their derivatives appearing in the governing equation remain bounded as $R \to \infty$, and that at least one scaled derivative of a dependent variable is $\mathcal{O}(1)$ (with the others being no larger). Admissible differential scalings are those that correspond to the desired $\mathcal{O}(1)$ variations in the scaled variables, although retaining a form of the governing equation that has at least two terms of dominant order of magnitude.

A scaling patch [3,4], therefore, employs a scaling, together with an interval of distances from the wall, in which the scaling under consideration is natural for the flow. Relevant to turbulent wall flows, this concept has similarities to Prandtl's and von Karman's original phenomenological notion of mixing lengths [6] and their domains of validity. That means that the derivatives of the scaled variables with respect to the scaled distance are $\mathcal{O}(1)$, that is, numerically bounded independent of the problem's small or large parameters. Examples, of course, are the traditional inner and outer domains. Automatically, the width of a scaling patch is at least as large as the characteristic length associated with the scaling.

A central aspect of the approach is based on a systematic method for locating scaling patches. Inherent to the method are the following, rational yet assumed, criteria for the determination of a scaling patch:

1) The proposed scaling must transform the MMB equation into an equation that still expresses a balance between forcelike quantities (which, of course, is what the original MMB equation does). This is the admissibility condition.

2) The proposed patch must be compatible with the flow, in that it can be shown rigorously by independent means that certain actual derivatives of the flow quantities at a position in the patch are of the order of magnitude implied by the scaling. This means that when scaled variables are used, those derivatives are bounded independently of R, and at least one derivative does not vanish, that is, $\mathcal{O}(1)$.

The procedure allows the systematic identification of the scaling patches using these criteria. For the wall-bounded turbulent flows considered by the authors thus far, there is a continuum of them parameterized by a parameter β which can with straightforward analysis be correlated with, for example, positions across the channel. This correlation is written in order of magnitude. Thus in an interval of distances from the wall, which stretches almost all the way across the flow domain, each location is the seat of a scaling patch within which the proper scaling of variables is known. As $\beta \to 0$, it is shown that the characteristic length in a given patch, which coincides in order of magnitude with the patch's width, is asymptotically proportional to its distance from the wall. This property is also reminiscent of mixing lengths, and constitutes a rigorous basis for the often hypothesized/assumed *distance from the wall* scaling [3], although also revealing the limitations of that hypothesis. With an additional reasonable assumption, it can also be shown that in any interior subinterval of the hierarchy defined in a definite way, the mean velocity profile approaches a logarithmic one as the Reynolds number approaches infinity [3]. Again, this result is similar to that obtained by arguments involving an assumed overlap of domains of validity of the inner and outer approximations, but its derivation is totally different.

Examples relating to both laminar and turbulent flows for large R are provided below. The equations for the laminar boundary layer are not underdetermined, and so it is not really necessary to employ the scaling patch procedure; but a scaling analysis of those equations is included for completeness. In the laminar flow example, the above criteria are shown to lead to the identification of a parameter-free normalization and subsequently recover the well-known similarity scaling. In this instance, the entire boundary layer can be viewed as encompassing a single scaling patch because a single scaling of the governing equation applies across the entire boundary layer. In the turbulent case, inner, meso, and outer normalizations are shown to reign over scaling patches that, respectively, cover only a portion of the flow. In either case, identification of the scaling patch requires invoking well-known physical behaviors (i.e., the vorticity is nonzero and decreasing with distance from the wall) or the examination of data to verify that the dominant terms in the governing equation are appropriately recovered. An essential aspect of the analysis involves use of the unintegrated form of the governing equation.

It may be argued that representations for the inner and the outer solutions are also valid in the region identified as the mesolayer, and therefore such a separate mesoscaling patch does not exist. This is not true; in fact, if one were to show such validity, and use an inner scaling, say, in the mesolayer, that scaling would render the mean velocity and Reynolds stress profiles at that location flat. This in turn would fail to provide the correct characteristic length in that region, which is in fact larger than that implied by inner scaling. The definition of a scaling patch specifically excludes such a possibility; it requires that variation of at least one of the statistical functions of interest be nonzero. This requirement is stated in the definition of "compatibility" discussed above. A similar argument excludes the possibility of the inner scaling patch covering the patches in the scaling hierarchy described below.

III. Results

Scaling patches for a number of flows are now identified, and in doing so, the associated mathematical and physical structure of the flow is revealed.

A. Laminar Boundary Layer Flow

The properties of the laminar boundary layer are established through the simultaneous action of advection and diffusion. As is well known (and derivable via a number of methods), these dynamical mechanisms underlie the classical $R^{1/2}$ scaling behavior. In the following, a well-known derivation for this scaling is provided that exemplifies elements that are also generic to the scaling patch approach.

Consider steady, laminar, two-dimensional, incompressible zero-pressure-gradient boundary layer flow in the x direction over a flat no-slip surface located at $y = 0$. Under these conditions, the momentum and continuity equations reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

The goal now is to find a normalization that appropriately captures the dynamical balance between advection and diffusion at all Reynolds numbers. [Note that this balance must exist everywhere in the flow because, even after the limiting process by which the boundary layer approximations are determined, (1) still contains two types of terms.] Such a normalization, of course, is one that renders (1) parameter free (i.e., the Reynolds number dependence becomes embedded within the normalization). With this aim in mind we fix a characteristic distance L down the plate; it specifies the region of interest. Define $u^* = u/U_\infty$ and $x^* = x/L$. To allow for the possibility of a parameter-free normalization of (1), normalizations for v and y are sought using the Reynolds number dependent forms, $v^* = R^n v/U_\infty$ and $y^* = R^m y/L$, where $R = U_\infty L/\nu$.

Normalization of (2) results in,

$$\frac{1}{R^{m-n}}\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{3}$$

and normalization of (1) yields,

$$u^*\frac{\partial u^*}{\partial x^*} + \frac{R^n}{R^m}v^*\frac{\partial u^*}{\partial y^*} = R^{2m-1}\frac{\partial^2 u^*}{\partial y^{*2}} \tag{4}$$

Because this system of equations is closed, to determine the values of m and n one may employ purely analytical methods [12,13]. In connection with the method of scaling patches and the indeterminate equations of wall turbulence, however, it is more relevant to illustrate how physical arguments that are, for example, based on easily obtainable empirical observations can be effectively employed to determine the underlying scaling behaviors.

Because the boundary layer approximations have already been invoked, the two terms in (3) must always be equal and opposite. This requires that $m = n$. Relative to (4) the following observations are made:

1) If $m > \frac{1}{2}$, then as $R \to \infty$ the equation reduces to $\partial^2 u^*/\partial y^{*2} \to 0$, or $\partial u^*/\partial y^* \to$ const.

2) If $m < \frac{1}{2}$, then as $R \to \infty$ one $\partial^2 u^*/\partial y^{*2}$ becomes negligibly small relative to the terms on the left.

The condition $m > \frac{1}{2}$ demands that as the Reynolds number become large as the shear stress approaches a constant. Physically, this possibility is rejected because it is contrary to the central notion that a boundary layer is a zone of nonzero vorticity, thus demanding the existence of a vorticity gradient across the layer. Furthermore, and perhaps at an even more fundamental level, the $m > \frac{1}{2}$ case also leads to a condition in which only one term is left in the momentum equation. This violates the condition of admissibility. Besides, it is physically highly questionable because the codominant terms in the boundary layer equation were already established in its derivation. Similarly, the $m < \frac{1}{2}$ case leads to an inviscid equation for the flow within the laminar boundary layer at high Reynolds number. This conclusion can also be summarily rejected because it is contrary to the irreducibility of viscous effects in the boundary layer. From these considerations it is thus concluded that $m = n = \frac{1}{2}$. Under this condition the governing equations become parameter free,

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{5}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \tag{6}$$

and the classical scalings, $v^* = R^{1/2}v/U_\infty$, $y^* = R^{1/2}y/L$ are recovered.

Important generic elements of the above analysis include, 1) the search for a parameter-free form of the momentum balance that continues to reflect the dominant dynamical mechanisms, and 2) the use of reasonable physical arguments (supported by empirical evidence) to guide the analysis. These elements are also central to the scaling patch methodology as applied to the underdetermined equations of wall turbulence. As shown, in the laminar flow case the entire flow is covered by a single scaling patch (i.e., one scaling is valid across the entire flow), whereas in turbulent flows multiple scaling patches may exist.

B. Prototypical Indeterminate Case: Turbulent Couette Flow

Elements of the scaling patch methodology as applied to a prototypical case of turbulent Couette flow are now provided. This is turbulent flow through a channel in which the forcing is provided not by a pressure gradient, but rather by the motion of the upper wall relative to the lower wall. Only a brief outline is provided here; a much more complete exposition can be found in the studies by Fife et al. [3,4].

The averaged equation of streamwise momentum balance for steady turbulent Couette flow expresses an exact balance between the transverse gradients of the viscous and Reynolds stresses. In inner variables, this equation is

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} = 0 \tag{7}$$

where T^+ is the inner-normalized Reynolds shear stress. There are also boundary conditions for the two unknowns U^+ and T^+. Physically, this equation has a level of analogy with the laminar boundary layer case just considered. That is, the momentum equation in the laminar case effectively describes a balance between axial advection and wall-normal diffusion. Similarly, the mean axial momentum equation in turbulent Couette flow describes the average balance between the wall-normal turbulent advection (recall that the origin of the Reynolds stress gradient term is the advective acceleration term in the Navier–Stokes equations) and wall-normal diffusion.

The analysis proceeds with small parameter $\epsilon^2 = \delta^{+-1}$. The outer length variable is $\eta = \epsilon^2 y^+$. The channel centerline is at $\eta = 1$. If we define a scaled Reynolds stress $\hat{T}(\eta)$ by

$$T^+ = T_m^+ + \epsilon^2\hat{T} \tag{8}$$

where $T_m^+ = T_{\eta=1}^+$ (value at the centerline), then the proper outer-normalized averaged momentum equation is seen to be

$$\frac{d^2U^+}{d\eta^2} + \frac{d\hat{T}}{d\eta} = 0 \tag{9}$$

In this regard it is significant to note that for Couette flow the "conventional" outer-normalized equation

$$\frac{1}{\delta^+}\frac{d^2U^+}{d\eta^2} + \frac{dT^+}{d\eta} = 0 \tag{10}$$

is ruled out because its form implies that neglecting the first term is appropriate because its nominal order of magnitude becomes small as $\delta^+ \to \infty$. Neglecting this term, however, violates the admissibility condition because it denies that the other term in the balance equation is, at every position, the same order of magnitude as the neglected term. One direct implication of this is that for Couette flow the origin of the well documented logarithmiclike behavior of the mean profile cannot be reasoned to exist via an overlap between a inner layer where, on average, viscous forces are important and an outer layer where, on average, viscous forces are negligible.

Equations (7) and (9) represent the appropriately normalized forms of the MMB for just two scaling patches. The MMB, however, rigorously admits a much richer scaling patch structure. Specifically, a continuous family of patches can be found by use of a family of "adjusted Reynolds stresses"

$$T^\beta(y^+) = T^+(y^+) - \beta y^+ \tag{11}$$

where β is a parameter whose range can be found from the properties of the function, $(dT^+/dy^+)(y^+)$. Specifically, because $(dT^+/dy^+)(y^+)$ vanishes at the wall and the centerline, and T^+ assumes positive values, then so does $(dT^+/dy^+)(y^+)$, and there is guaranteed to exist an interval where $(dT^+/dy^+)(y^+)$ decreases from a local (maybe global) maximum to 0. The range of the function, $(dT^+/dy^+)(y^+)$, in that interval is very nearly the range of values of β. The actual numbers delineating this range can be found from empirical data.

This is also the set of values of β for which the graph of $T^\beta(y^+)$ has a local maximum at some location $y^+ = y_m^\beta$. It can readily be shown using the scaling patch criteria listed above that there is a scaling patch approximately centered on that maximum, and the characteristic length in that patch is $\mathcal{O}(\beta^{-1/2})$. The patch with the least value of β, namely $\beta = \epsilon^4$, is essentially identical to the outer scaling domain.

Further analysis allows one to correlate values of β with locations in the channel, and to determine many qualitative properties of the profiles $U^+(y^+), T^+(y^+)$. For example, important information about the classical question of the existence of logarithmiclike profiles for U^+ is obtained. This includes specifying the region of the flow over which logarithmiclike behavior will exist, an MMB based reason for why the profile might have two logarithmiclike regions of slightly different slope, and the conditions on the scaled second derivative of the Reynolds stress that must be met in order for logarithmic behavior to exactly be realized [1,3]. Also, the sense in which the characteristic length of each scaling patch in the hierarchy is or is not proportional to distance from the wall, that is, to y_m^β, is clarified [3].

C. Extended Examples

Pure turbulent Couette flow is solely composed of a layer whose dynamical balance is between the mean viscous force and mean turbulent momentum transfer. Wei et al. [1] identify this as a *stress gradient balance layer*, and show that such layers also exist in boundary layer, pipe, and channel flows. They further show that in pressure driven channel flow, for example, there is also an authentic outer layer in the sense that the dominant terms in the momentum balance are the Reynolds stress gradient and mean axial pressure gradient (i.e., are inviscid mechanisms). In connection with this, they show that in moving from the stress gradient balance layer to the outer layer a mathematical *balance breaking and exchange* process occurs. This process is identical in structure to that which occurs between the different layers in the hierarchy described above. In the case of pipe and channel flows, Wei et al. [2] effectively use the properties of the intermediate scaling patch that exists between the stress gradient balance layer and the outer layer (layers II and IV in their nomenclature) to scale (mesoscale) the Reynolds stress in the interior zone [of size $\mathcal{O}(\delta^{+1/2})$] where both inner and outer normalizations fail to merge the differing Reynolds number data. (Some planar Poiseuille flow Reynolds stress data are shown in Fig. 2 below.) This scaling is analytically derived via the process of finding a parameter-free representation of the MMB that satisfies the admissibility and compatibility conditions described above.

A common element of all the cases discussed thus far, however, is that the scaling is related only to the inverse Reynolds number small parameter. Efforts to extend the scaling patch method to more complicated cases are ongoing. Specifically, the turbulent Couette–Poiseuille flow and channel flow heat transfer problems discussed below are characterized by parameters representing multiple physical effects, whereas the sink-flow boundary layers have additional applied force relative to the zero-pressure-gradient case. In Couette–Poiseuille flow the additional effect is the imbalance of the shear stesses at the upper and lower walls owing to the relative wall motion. Similarly, in fully developed turbulent heat transfer in a channel there are the simultaneous effects of Prandtl and Reynolds number, whereas in the favorable pressure gradient boundary layer the force balance effectively has an additional term. In what follows, these extensions to the method are briefly discussed.

Turbulent Couette–Poiseuille Flow: This is the case of fully developed two-dimensional (in the mean), turbulent flow in a channel that is simultaneously driven by a mean axial pressure gradient and a relative wall motion in the plane of the wall. The mathematical formulation of this flow is aided by the fact that, owing to symmetry considerations, one can always formulate the problem such that the pressure gradient generating flow is in the positive x direction and the upper wall moving. Wei et al. [14] show that the innernormalized MMB for this flow can be expressed as

$$\frac{d^2U^+}{dy^{+2}} + \frac{dT^+}{dy^+} + \epsilon^2\omega = 0 \tag{12}$$

where ϵ is defined as previously (using the friction velocity at the lower wall), and $0 \le \omega \le 1$, depending on the Couette and Poiseuille contributions. Relative to the balance breaking and exchange process that occurs in pure Poiseuille flow, the mesoscaling is modified owing to the appearance of ω. An analogous analysis can be constructed, however, and the results in Fig. 2 compare the traditional inner and outer normalizations with the mesonormalization. As can be seen from a comparison of the pure Poiseuille data of Figs. 2a and 2b, neither inner nor outer normalizations scale the data in the vicinity of the peak. Figure 2c shows that the scaling obtained from the analysis convincingly merges available C-P flow Reynolds stress data over a considerable interior portion of the flow including the region around the peak (meso and outer layers; the mesoscaling for the Reynolds stress naturally extends to the outer layer because $d\hat{T}/d\hat{y} = dT^+/d\eta$). The portions of the profiles near the two walls do not follow the mesoscaling because they reside in separate patches that, as might be expected, adhere to an inner scaling based on the friction velocity associated with the given wall. Evidence of this for the patch near the lower wall is given in Fig. 2a. Once again, it is relevant to note that the scaling patches revealed in Fig. 2 are not postulated but are explicitly determined from an analysis of the MMB in accordance with the scaling patch criteria. Specifically, determining the mesoscaling does not require: curve fitting, postulating the existence of an overlap layer, use of composite or other asymptotic expansions, invoking a similarity hypothesis or a reliance on the existence/nonexistence of a logarithmic mean velocity profile. It is simply the scaling admitted by the MMB in conceptually the same way that the $R^{1/2}$ scaling renders the solutions to the laminar boundary layer equations invariant. As with Poiseuille flow alone [2], an approximate mesoscaling is also possible (Fig. 2d).

Fully Developed Heat Transfer in a Channel: Wei et al. [15] consider fully developed turbulent heat transfer in a channel owing the presence of a constant wall heat flux. In this case the relevant transport equation is the thermal energy balance, which, when appropriately expressed, attains a form similar to that of the innernormalized MMB for turbulent Couette–Poiseuille flow,

$$\frac{d^2\Psi}{dy_\sigma^2} + \frac{dT_\theta}{dy_\sigma} + \sigma^2 r_\sigma = 0 \tag{13}$$

In this equation y_σ is a modified innernormalized wall-normal distance, and $r_\sigma = \mathcal{O}(1)$ over most of the channel. As in the case of Couette–Poiseuille flow, the last term contains two components. In this case, however, the problem is mathematically richer because r_σ is a function of y_σ. A modified multiscale analysis can still be conducted, and the results of Fig. 3 use available data to compare the traditional inner normalization with the resulting mesonormalization. As with the C-P flow Reynolds stress data, the mesonormalized turbulent heat flux profiles are shown to merge to a single profile, except near the wall where a modified inner scaling patch can be shown to exist. With regard to the mesoscaled profile, it is also worth noting that the analysis predicts that this normalization will lose validity for sufficiently low Peclet number. Thus, the observed deviation of the $\delta^+ = 180$, $Pr = 0.1$ profile is expected.

Boundary Layers with Favorable Pressure Gradient: In [16] a scaling patch analysis is given for a turbulent boundary layer with imposed pressure gradient that is favorable, that is, negative. The prototypical example is when the core flow is directed toward a sink. The basic innernormalized mean momentum balance is like (12), but with ω replaced by a function $r(y^+)$ whose qualitative properties are simple and known, and ϵ has a different meaning, involving the acceleration parameter. The quantities r and ϵ also depend on the horizontal coordinate except in the region of fully developed sink flow; however, in any case, the methodology given herein was also successfully used to determine the scaling structure of steady flows.

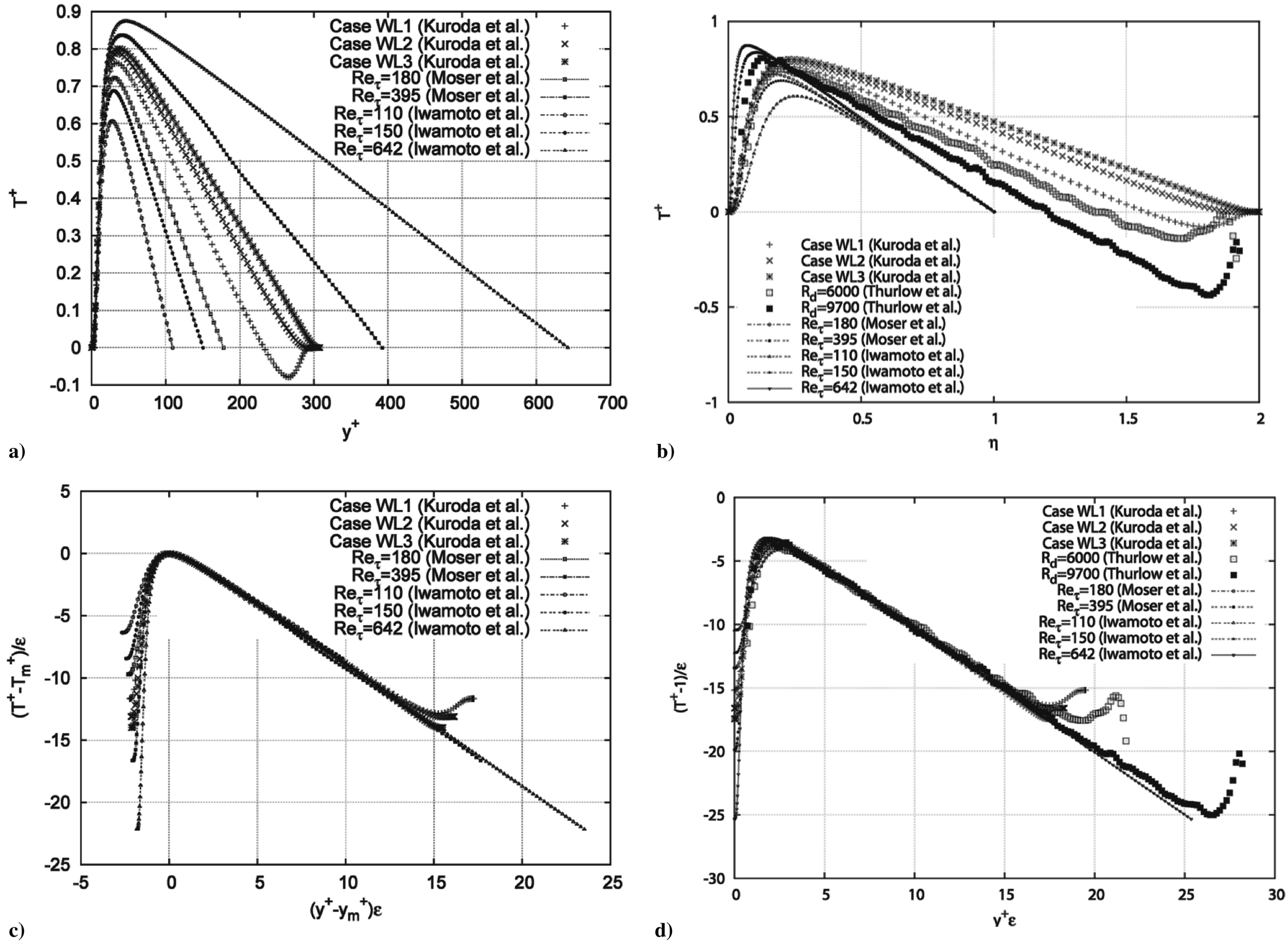


Fig. 2 Normalizations of the Reynolds shear stress in turbulent Couette–Poiseuille flow: a) inner, b) outer, c) meso, d) approximate meso [14].

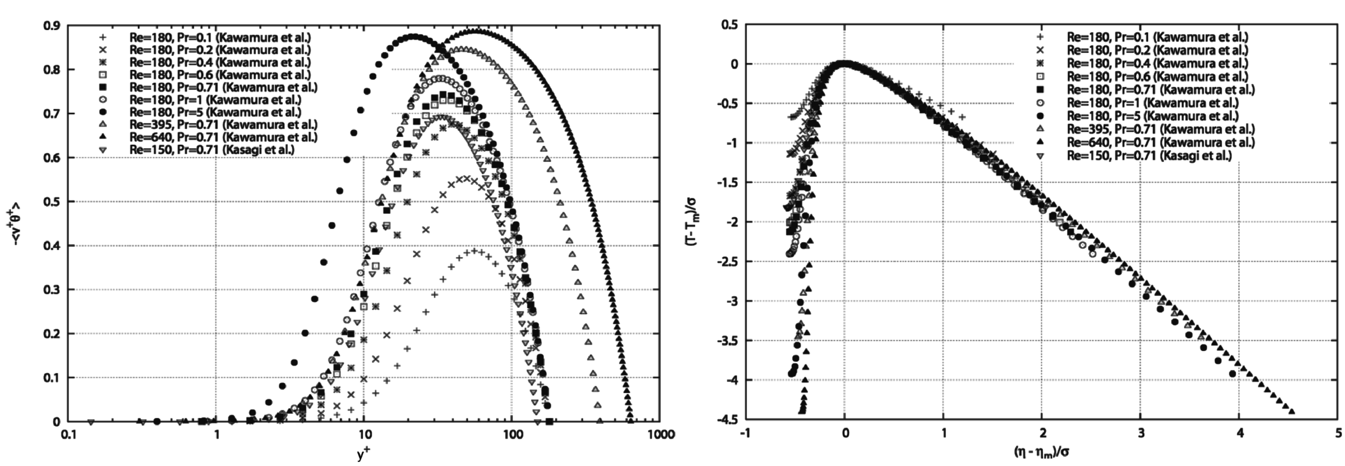


Fig. 3 Innernormalized a) and mesonormalized b) turbulent heat flux profiles in fully developed channel flow.

IV. Conclusion

This paper provides an overview of the method of scaling patches as applied to the underdetermined equations of wall turbulence. The method employs generic notions that are well established relative to scaling the equations of fluid dynamics, or, for that matter any other differential equation. Central among these is the search for transformations of the appropriate form of the balance equation that render the equation invariant with changes in the relevant parameter (s). As exemplified by the laminar boundary layer problem of Sec. III.A (and numerous other problems studied during the 20th century [17]), such transformations inherently reveal the universal scalings for the solutions to these equations, independently of whether these solutions can be written in closed form or, for example, only approximated via asymptotic expansions.

Arguably, however, this notion has previously not been sufficiently formalized for handling indeterminate equations and thus has been underused relative to determining the scaling behaviors of the Reynolds averaged Navier–Stokes (RANS) equations.

Specifically, under either the similarity or overlap layer approaches discussed at the outset once a particular normalization is hypothesized as a scaling, the MMB no longer comes into play. Instead the analysis focuses on postulating certain types of approximate solution techniques (e.g., composite expansions) and empirically exploring the consequences of these postulated solutions. In contrast, the present methodology focuses on clearly identifying and verifying the conditions that must be met in order for the MMB to admit any given scaling, and, once established, employing the MMB itself to discern the consequences. As noted previously, empirical data are used to guide the analysis generally by verifying the order of magnitude of various terms. The examples provided herein show that the method can reveal the natural scalings over their associated domains (patches) directly from analysis of the RANS equations. By doing so, the method has also generated a picture of turbulent wall-flow structure and physics that is a considerable departure from the predominant view [1,3,5], although remaining compatible with known results. Ongoing efforts continue to extend the methodology.

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References

[1] Wei, T., Fife, P., Klewicki, J., and McMurtry, P., "Properties of the Mean Momentum Balance in Turbulent Boundary Layer, Pipe and Channel Flows," *Journal of Fluid Mechanics*, Vol. 522, Jan. 2005, pp. 303–327.

[2] Wei, T., McMurtry, P., Klewicki, J., and Fife, P., "Mesoscaling of the Reynolds Shear Stress in Turbulent Channel and Pipe Flows," *AIAA Journal*, Vol. 43, No. 11, 2005, pp. 2350–2353.

[3] Fife, P., Wei, T., Klewicki, J., and McMurtry, P., "Stress Gradient Balance Layers and Scale Hierarchies in Wall Bounded Turbulent Flows," *Journal of Fluid Mechanics*, Vol. 532, May 2005, pp. 165–189.

[4] Fife, P., Klewicki, J., McMurtry, P., and Wei, T., "Multiscaling in the Presence of Indeterminacy: Wall-Induced Turbulence," *Multiscale Modeling and Simulation*, Vol. 4, No. 3, 2005, pp. 936–959.

[5] Klewicki, J., McMurtry, P., Fife, P., and Wei, T., "A Physical Model of the Turbulent Boundary Layer Consonant with the Dynamical Structure of the Mean Momentum Balance," in *Proceedings of the 15th Australasian Fluid Mechanics Conference*, The University of Sydney, Sydney, Australia, Paper AFMC00058, 2004.

[6] Prandtl, L., "Bericht uber die Entstehung der Turbulenz," *Zeitschrift fur Angewandte Mathematik und Mechanik*, Vol. 5, No. 1, 1925, pp. 136–139.

[7] von Karman, T., "Mechanical Similarities in Turbulence," NACA-TM-611, 1931.

[8] Barenblatt, G., and Chorin, A., "A Mathematical Model for the Scaling of Turbulence," *Proceedings of the National Academy of Sciences*, Vol. 101, No. 42, 2004, pp. 15023–15026.

[9] Barenblatt, G., and Chorin, A., "New Perspectives in Turbulence: Scaling Laws, Asymptotics, and Intermittency," *Society of Industrial and Applied Mathematics Review*, Vol. 40, No. 2, 1998, pp. 265–291.

[10] Barenblatt, G., Chorin, A., and Prostokishin, V., "A Model of a Turbulent Boundary Layer with a Nonzero Pressure Gradient," *Proceedings of the National Academy of Sciences*, Vol. 99, No. 9, 2002, pp. 5772–5776.

[11] Millikan, C., "A Critical Discussion of Turbulent Flows in Channels and Circular Tubes," *Proceedings of the Fifth International Congress of Applied Mechanics*, John Wiley Inc., New York, NY, 1939, pp. 386–392.

[12] Hansen, A., *Similarity Analyses of Boundary Value Problems in Engineering*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.

[13] Cantwell, B., *Introduction to Symmetry Analysis*, Cambridge University Press, Cambridge, 2002.

[14] Wei, T., Fife, P., and Klewicki, J., "On Scaling the Mean Momentum Balance and Its Solutions in Turbulent Couette–Poiseuille Flow," *Journal of Fluid Mechanics* (to be published).

[15] Wei, T., Fife, P., Klewicki, J., and McMurtry, P., "Scaling Heat Transfer in Fully Developed Turbulent Channel Flow," *International Journal of Heat and Mass Transfer*, Vol. 48, December 2005, pp. 5284–5296.

[16] Metzger, M., and Fife, P., "Scaling Properties of Turbulent Boundary Layers with Favorable Pressure Gradient" *Journal of Fluid Mechanics* (submitted for publication).

[17] Schlichting, H., and Gersten, K., *Boundary Layer Theory*, Springer–Verlag, Inc., Berlin, 2000.

[18] Klewicki, J., and Falco, R., "On Accurately Measuring Statistics Associated with Small Scale Structure in Turbulent Boundary Layers Using Hot-Wire Probes," *Journal of Fluid Mechanics*, Vol. 219, January 1990, pp. 119–142.

[19] Kuroda, A., Kasagi, N., and Hirata, M., "A Direct Numerical Simulation of the Turbulent Flow Between Two Parallel Walls: Turbulence Characteristics near the Wall Without Mean Shear," in *Proceedings of the 5th Symposium on Numerical Simulation of Turbulence*, IIS of the University of Tokyo, Tokyo, 1990, pp. 5–11.

[20] Moser, R., Kim, J., and Mansour, N., "Direct Numerical Simulation of Turbulent Channel Flow up to $R_\tau = 590$" *Physics of Fluids*, Vol. 11, No. 4, 1999, pp. 943–954.

[21] Imwamoto, K., Suzuki, Y., and Kasagi, N., "Reynolds Number Effect on Wall Turbulence: Toward Effective Feedback Control," *International Journal of Heat and Fluid Flow*, Vol. 23, No. 7, 2002, pp. 678–689.

[22] Thurlow, E., and Klewicki, J., "Experimental Study of Turbulent Poiseuille–Couette Flow," *Physics of Fluids*, Vol. 12, No. 4, 2000, pp. 865–875.

[23] Kawamura, H., Abe, H., and Shingai, K., "DNS of Turbulence and Heat Transport in a Channel Flow with Different Reynolds and Prandtl Numbers and Boundary Conditions," in *Turbulence, Heat and Mass Transfer 3: Proceedings of the 3rd International Symposium of Turbulence, Heat and Mass Transfer*, Aichi Shuppan, Tokyo, 2000, pp. 15–32.

[24] Kasagi, N., Tomita, Y., and Kuroda, A., "Direct Numerical Simulation of the Passive Scalar Field in A Two-Dimensional Turbulent Channel Flow," in *Proceedings of the 3rd American Society of Mechanical Engineers/Japanese Society of Mechanical Engineers Thermal Engineering Joint Conference*, American Society of Mechanical Engineers, New York, 1991, pp. 175–182.

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